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An elementary proof of the Ambartzumian–Pleijel identity

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1. *Introduction*

In [**5**], Pleijel proved an identity relating the area A of a convex plane domain and the length L of its boundary (of class C^1). In particular, it contains the isoperimetric inequality $L^2 - 4\pi A \geqslant 0$.

Ambartzumian gave two proofs of a generalized version of the Pleijel-identity for convex polygons. The first proof (in [**1**]) consisted of direct computations. In his book [**2**] however, he shows that the identity is an easy consequence of the solution to the Buffon–Sylvester problem.

Pohl proved an analogous formula for closed convex plane curves with smooth boundary, applying Stokes' theorem to a suitable manifold with boundary (see [**6**]).

The aim of this note is to show that Stokes' theorem may also be used to prove Ambartzumian's Pleijel-type identity for convex polygons directly. It turns out that the use of differential forms leads to considerable simplifications. The interesting question whether this method may be used to derive a Pleijel-type identity for more general convex domains, remains unanswered.

2. *Ambartzumian's Pleijel-type identity for convex polygons*

Throughout this section, let C denote a (bounded) closed convex polygon in the plane. The main idea of the proof is to compute the integral of a differential form over two of the sides of C. Then by a limiting procedure the result follows immediately.

To be able to perform the integration, we have to give an orientation to the sides.

Let a and b be two non-intersecting sides of C that do not share any of their endpoints A_1, A_2 and B_1, B_2 respectively. The set $a \times b$ is a two-dimensional submanifold of $\mathbb{R}^4$, which can be parametrized in the following way. Let u and v be the vectors $A_2 - A_1$ and $B_2 - B_1$ respectively. Then $x \in a$ and $y \in b$ have the representation

$$x = A_1 + \theta_1 u, \quad y = B_1 + \theta_2 v,$$

for some numbers $\theta_1, \theta_2 \in [0, 1]$.

If dl_1 (resp. dl_2) is the element of length in a (resp. b), directed from A_1 to A_2 (resp. B_1 to B_2), then the 2-form $dl_1 \wedge dl_2$ has the representation

$$dl_1 \wedge dl_2 = |a| . |b| \, d\theta_1 \wedge d\theta_2,$$

where $d\theta_1 \wedge d\theta_2$ is the canonical 2-form on $\mathbb{R}^2$ and $|x|$ denotes the length of the side x.

Using this parametrization, we can consider $a \times b$ as an oriented manifold with boundary. Define the mapping $\phi : [0, 1]^2 \rightarrow a \times b$ by

$$\phi(\theta_1, \theta_2) = (A_1 + \theta_1 u, B_1 + \theta_2 v).$$

Then we have

$$a \times b = \phi([0,1]^2)$$

and the oriented boundary of $a \times b$ is identified by this mapping with the boundary in $\mathbb{R}^2$ of the unit square with the usual counter-clockwise orientation. From this identification, it is seen that $a \times \{B_1\}$ and $\{A_2\} \times b$ have the same orientation as a, b respectively, and that $a \times \{B_2\}$ and $\{A_1\} \times b$ have the opposite orientation.

We shall need the following lemma in the proof.

LEMMA. *Let a and b be as described above and let (x, y) be a point on $a \times b$. Let dl_1, dl_2 denote the element of length on a and b respectively and let χ denote the length of the segment joining x and y, that is directed from x to y. Furthermore, let α_1 and α_2 be the angles, lying to the right of χ, formed by χ and the sides a and b respectively.*

Then we have, for fixed y

$$d\alpha_1 = \frac{\sin \alpha_2}{\chi} dl_2 \tag{1}$$

and for fixed x

$$d\alpha_2 = -\frac{\sin \alpha_1}{\chi} dl_1. \tag{2}$$

Proof. First fix l_1. Let h_x be the length of the perpendicular from x onto b. Then

$$\frac{h_x}{-l_2} = \tan(\pi - \alpha_2) = -\tan \alpha_2$$

hence

$$\alpha_2 = \arctan \frac{h_x}{l_2}.$$

Consequently

$$\frac{d\alpha_2}{dl_2} = -\frac{h_x}{l_2^2 + h_x^2} = -\frac{h_x}{\chi^2} = -\frac{\sin \alpha_2}{\chi}.$$

Since clearly $\alpha_1 + \alpha_2$ is constant, the first assertion follows.

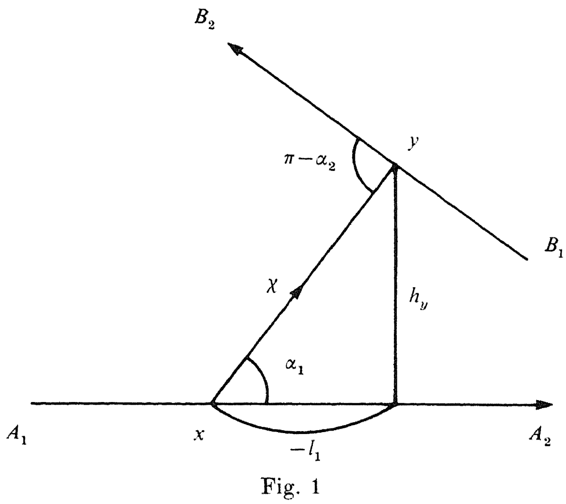

Fig. 1

Next, fix l_2 and let h_y be defined similarly to h_x. Then

$$\frac{d\alpha_2}{dl_1} = -\frac{d\alpha_1}{dl_1} = -\frac{d}{dl_1}\arctan\left(-\frac{h_y}{l_1}\right) = -\frac{\sin\alpha_1}{\chi}.$$

This proves the lemma.

Observe that if l_2 increases, for l_1 fixed, then the angle α_1 increases. On the other hand, if l_1 increases, for l_2 fixed, then the angle α_2 decreases. As a consequence, we see that the signs of (1) and (2) are correct.

We are now ready to prove the Pleijel-type identity.

THEOREM (Ambartzumian–Pleijel). *Let C be a convex polygon with n sides a_i of length $|a_i|$. Suppose that C is oriented as described above. Let $f:\mathbb{R}\to\mathbb{R}$ be a C^1-function. Then*

$$\int_{[C]} f(\chi)\,dg = \int_{[C]} f'(\chi)\,\chi\cot\alpha_1\cot\alpha_2\,dg + \sum_{i=1}^{n}\int_0^{|a_i|} f(x)\,dx,$$

where dg denotes the element of an invariant measure on the set G of non-oriented lines in the plane and $[C] := \{g\in G : g\cap C \neq \varnothing\}$.

Proof. First consider two sides a and b with endpoints A_1, A_2 and B_1, B_2 respectively. Suppose that a and b are non-intersecting but not parallel and that they do not share an endpoint.

Consider the orientation-preserving differential form $dl_1\wedge dl_2$, where dl_1 (resp. dl_2) is the element of length along a (resp. b), as defined above. Define the 1-form ω on $a\times b$ by

$$\omega(x,y) = \cos\alpha_1\,dl_1 + \cos\alpha_2\,dl_2.$$

Then

$$d\omega = -\sin\alpha_1\,d\alpha_1\wedge dl_1 - \sin\alpha_2\,d\alpha_2\wedge dl_2. \tag{3}$$

By the Lemma, (3) may be written as

$$d\omega = -\frac{\sin\alpha_2}{\chi}\sin\alpha_1\,dl_2\wedge dl_1 + \frac{\sin\alpha_1}{\chi}\sin\alpha_2\,dl_1\wedge dl_2$$

whence, by the anti-commutativity of the exterior product

$$d\omega = 2\frac{\sin\alpha_1\sin\alpha_2}{\chi}\,dl_1\wedge dl_2. \tag{4}$$

Define $\omega_1 := f(\chi)\,\omega$. Then we may apply Stokes' theorem (see e.g. [**4**]) to the 1-form ω_1 on $a\times b$, since the latter is an oriented 2-manifold with boundary. This yields

$$\int_{\partial(a\times b)} \omega_1 = \int_{a\times b} d\omega_1 = \int_{a\times b} f'(\chi)\,d\chi\wedge\omega + \int_{a\times b} f(\chi)\,d\omega.$$

Observe that

$$\frac{-d\chi}{dl_2} = \cos(\pi-\alpha_2) = -\cos\alpha_2,$$

hence

$$d\chi = \cos\alpha_2\,dl_2.$$

Analogously, we have

$$d\chi = -\cos\alpha_1\,dl_1.$$

Consequently

$$d\chi\wedge\omega = d\chi\wedge\cos\alpha_1\,dl_1 + d\chi\wedge\cos\alpha_2\,dl_2 = -2\cos\alpha_1\cos\alpha_2\,dl_1\wedge dl_2.$$

By (4),
$$\int_{a\times b} f(\chi)\,d\omega = 2\int_{a\times b} \frac{\sin\alpha_1 \sin\alpha_2}{\chi}\, dl_1 \wedge dl_2.$$

Hence (5) may be written as

$$\int_{a\times b} f(\chi)\frac{\sin\alpha_1 \sin\alpha_2}{\chi}\, dl_1 \wedge dl_2 = \int_{a\times b} \cos\alpha_1 \cos\alpha_2 f'(\chi)\, dl_1 \wedge dl_2 + \frac{1}{2}\int_{\partial(a\times b)} f(\chi)\,\omega. \quad (6)$$

At the beginning of the section, we showed that the boundary $\partial(a\times b)$ of $a\times b$ is

$$\bigcup_{i=1,2} (a\times\{B_i\}) \cup \bigcup_{i=1,2} (\{A_i\}\times b).$$

Consequently

$$\left.\begin{aligned} \int_{\partial(a\times b)} f(\chi)\,\omega &= \int_{\{A_1\}\times b} f(\chi)\,w + \int_{\{A_2\}\times b} f(\chi)\,\omega + \int_{a\times\{B_1\}} f(\chi)\,\omega + \int_{a\times\{B_2\}} f(\chi)\,\omega \\ &= -\int_{\{A_1\}\times b} f(\chi)\cos\alpha_2\, dl_2 + \int_{\{A_2\}\times b} f(\chi)\cos\alpha_2\, dl_2 \\ &\quad + \int_{a\times\{B_1\}} f(\chi)\cos\alpha_1\, dl_1 - \int_{a\times\{B_2\}} f(\chi)\cos\alpha_1\, dl_1, \end{aligned}\right\} \quad (7)$$

where one has to take the orientation into consideration. Equation (7) corresponds to equation (21) in [1], in a version for directed lines.

Next, we let the endpoint B_1 of b tend to the endpoint A_2 of a, i.e. the distance between B_1 and A_2 tends to zero. Then in the limit, where $A_2 = B_1$, we get

$$\int_{\partial(a\times b)} f(\chi)\,\omega = \int_0^{|a|} f(x)\,dx + \int_0^{|b|} f(x)\,dx - \int_{a\times\{B_2\}} f(\chi)\cos\alpha_1\, dl_1 - \int_{\{A_1\}\times b} f(\chi)\cos\alpha_2\, dl_2. \quad (8)$$

Summation of (6) over all sides of C, using (8) as well as the lemma, completes the proof of the theorem. Observe that indeed terms of the form $\int_0^{|a_i|} f(x)\,dx$ appear twice in the sum. Furthermore, there is cancellation of terms of the form

$$\int_{\{A_i\}\times a_j} f(\chi)\cos\alpha_2\, dl_2 \quad \text{and} \quad \int_{a_i\times\{A_j\}} f(\chi)\cos\alpha_1\, dl_1$$

as desired.

This note is a version of one of the sections of the author's master's thesis [3]. The author would like to thank P. Groeneboom again, under whose supervision she had the pleasure of writing it.

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